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Published in:
Proceedings of the European Control Conference, 2007

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2007

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Schaft, A. V. D. (2007). On balancing of passive systems. In *Proceedings of the European Control Conference, 2007* University of Groningen, Johann Bernoulli Institute for Mathematics and Computer Science.

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On balancing of passive systems

Arjan van der Schaft

Abstract—It is shown how the application of various standard balancing techniques to general lossless systems basically leads to the same result: the pair of to-be-balanced functions is given by two copies of the energy function. Hence balancing will not yield any information about the relative importance of the state components in a balanced realization. This result is extended to the lossy case, indicating that balancing in this case will largely depend on the internal energy dissipation. By using the representation of passive systems as port-Hamiltonian systems a direction for extending standard balancing is discussed.

Keywords: Balancing, scattering, passive systems, port-Hamiltonian systems, modal analysis.

I. INTRODUCTION

Modeling of technological or physical systems often leads to high-dimensional dynamical models. The same occurs if distributed-parameter models are spatially discretized. An important issue concerns model reduction of these high-dimensional systems, both for analysis and control.

Within the systems and control literature a popular and elegant tool for model reduction is *balancing*, dating back to [8]. One favorable property of model reduction based on balancing, as compared with other techniques such as modal analysis, is that the approximation of the dynamical system is explicitly based on its input-output properties.

In Section III we investigate various balancing approaches for general (linear and nonlinear) dynamical systems that are *lossless*, i.e., *passive* without internal energy dissipation. It is shown that standard balancing in the lossless case does not yield information about the relative importance of state components in a balanced representation, while in the passive case (Section IV) it is largely determined by the internal energy dissipation, which may not be a desirable feature. This suggests to look for more refined balancing techniques, which is the subject for ongoing research discussed in Section V.

II. LOSSLESS SYSTEMS

Consider the square nonlinear input-state-output system

$$\begin{aligned} \dot{x} &= a(x) + b(x)u \\ \Sigma : \quad y &= c(x) + d(x)u \end{aligned} \quad (1)$$

where $u, y \in \mathbb{R}^m$, and $x \in \mathbb{R}^n$ are local coordinates for an n -dimensional state space manifold \mathcal{X} . In such local coordinates $a(x)$ denotes an n -dimensional vector, $b(x)$ an

$n \times m$ -dimensional matrix, $c(x)$ an m -dimensional vector, while $d(x)$ is an $m \times m$ -dimensional matrix. Throughout we assume the existence of a distinguished *equilibrium* x_0 , that is,

$$a(x_0) = 0, \quad c(x_0) = 0 \quad (2)$$

The system Σ is called *lossless* [16] if there exists a *storage function* $H : \mathcal{X} \rightarrow \mathbb{R}$ with $H(x_0) = 0$ and $H(x) \geq 0$ for every $x \neq x_0$, such that

$$H(x(t_2)) - H(x(t_1)) = \int_{t_1}^{t_2} u^T(t)y(t)dt \quad (3)$$

for all solution trajectories $(u(\cdot), x(\cdot), y(\cdot))$ of the system Σ and all time instants $t_1 \leq t_2$. The system is *passive* if the equality in (3) is replaced by the inequality \leq .

Remark 2.1: The assumption $H(x_0) = 0$ is not essential (see [16], [11] for the general case), but will be made for simplicity throughout this paper.

Lossless systems are abundant in physical modeling by equating H with the energy stored in the physical system and $u^T y$ with the power supplied to the system; at least if we make the idealizing assumption that there is no internal energy dissipation in the system (and if there is internal dissipation then the system is passive). Henceforth we call the inputs u and outputs y the *power variables*.

It is well-known that if the function H is *differentiable* then the property of being lossless is equivalent to:

$$\begin{aligned} \frac{\partial^T H}{\partial x}(x)a(x) &= 0 \\ c(x) &= b^T(x) \frac{\partial H}{\partial x}(x) \\ d(x) &= -d^T(x) \end{aligned} \quad (4)$$

where $\frac{\partial H}{\partial x}(x)$ denotes the n -dimensional column vector of partial derivatives of H . If additionally H is continuously differentiable and *positive definite*, that is, $H(x) > 0$ for every $x \neq x_0$, then it immediately follows that the equilibrium x_0 is *stable*, but *not* asymptotically stable, since H is a Lyapunov function that is constant along solution trajectories.

Remark 2.2: The *linear* system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (5)$$

with distinguished equilibrium $x_0 = 0$ is lossless if there exists a *quadratic* storage function $H(x) = \frac{1}{2}x^T Qx$ with $Q = Q^T \geq 0$ satisfying

$$A^T Q + QA = 0, \quad C = B^T Q, \quad D = -D^T \quad (6)$$

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In order to relate balancing to physical energy considerations it turns out to be useful to switch to the so-called *scattering representation* Σ_s of Σ , which is obtained by the following transformation of the external variables (inputs and outputs) u and y :

$$\begin{aligned} v &= \frac{1}{\sqrt{2}}(u + y) \\ z &= \frac{1}{\sqrt{2}}(-u + y) \end{aligned} \quad (7)$$

with inverse

$$\begin{aligned} u &= \frac{1}{\sqrt{2}}(v - z) \\ y &= \frac{1}{\sqrt{2}}(v + z) \end{aligned} \quad (8)$$

Substitution of these last expressions into Σ with $d(x) = 0$ yields the scattering representation Σ_s [11]

$$\begin{aligned} \dot{x} &= a(x) - b(x)c(x) + \sqrt{2}b(x)v \\ \Sigma_s : \quad z &= \sqrt{2}c(x) - v \end{aligned} \quad (9)$$

which can be regarded as an input-state-output system with input v (the 'incoming wave') and output z (the 'outgoing wave').

Remark 2.3: Similar, but more involved, formulas can be derived for the case $d \neq 0$ under the assumption that the matrix $I - d(x)$ is invertible.

Remark 2.4: An analogous representation follows by considering z to be the input and v the output (while for the case $d \neq 0$ the assumption is made that the matrix $I + d(x)$ is invertible).

We collect the following equalities relating the power variables u, y with the wave variables v, z :

$$\begin{aligned} \frac{1}{2} \|v\|^2 - \frac{1}{2} \|z\|^2 &= u^T y \\ \frac{1}{2} \|u\|^2 + \frac{1}{2} \|y\|^2 &= \|z\|^2 + u^T y = \|v\|^2 - u^T y \\ \|v\|^2 + \|z\|^2 &= \|u\|^2 + \|y\|^2 \\ &\quad (\text{parallelogram identity}) \end{aligned} \quad (10)$$

The first equality represents the basic relation between the power variables u, y and the wave variables v, z . Indeed, using the first equality we obtain the following equivalent characterization of losslessness in terms of the scattering representation Σ_s :

$$H(x(t_2)) - H(x(t_1)) = \int_{t_1}^{t_2} \frac{1}{2} \|v(t)\|^2 - \frac{1}{2} \|z(t)\|^2 dt \quad (11)$$

for all solution trajectories $(v(\cdot), x(\cdot), z(\cdot))$ of the system Σ_s and all time instants $t_1 \leq t_2$. Note that we may regard the term $\frac{1}{2} \|v(t)\|^2$ as the incoming power associated to the incoming wave v , and $\frac{1}{2} \|z(t)\|^2$ as the outgoing power corresponding to the outgoing wave z .

While the equilibrium x_0 of Σ for $u = 0$ is only *stable* it is under quite general conditions an *asymptotically stable* equilibrium for Σ_s with $v = 0$. Indeed, if H is continuously differentiable and positive definite then we obtain for $v = 0$

$$\frac{d}{dt} H(x(t)) = -\frac{1}{2} \|z(t)\|^2 = -\|y(t)\|^2 \quad (12)$$

ensuring asymptotic stability if the system Σ_s for $v = 0$ is *zero-state detectable* (with x_0 representing the zero-state) [11]. Similarly, the *time-reversed* system Σ_s for $z = 0$ satisfies

$$\frac{d}{d(-t)} H(x(t)) = -\frac{1}{2} \|v(t)\|^2 = -\|y(t)\|^2 \quad (13)$$

This motivates the following standing assumption.

Assumption 2.1: The equilibrium x_0 is globally asymptotically stable for Σ_s with $v = 0$, and globally asymptotically stable for the time-reversed system Σ_s with $z = 0$.

Remark 2.5: For a linear system Σ given by (5) with $D = 0$ the scattering representation (9) reduces to

$$\begin{aligned} \dot{x} &= (A - BC)x + \sqrt{2}Bv \\ z &= \sqrt{2}Cx - v \end{aligned} \quad (14)$$

Hence, Assumption 2.1 is satisfied if and only if the pair (C, A) is *detectable*.

III. BALANCING OF LOSSLESS SYSTEMS

In this section we will show how various balancing procedures for lossless systems all lead to more or less the same answer: all state components in a balanced representation are equally important.

A. The power variable representation

We start with the system representation Σ with power variables u, y . Since the equilibrium x_0 is stable but *not* asymptotically stable, we cannot apply ordinary balancing. Instead we consider LQG-balancing [6], or more precisely, its extension to the nonlinear case using past and future energies, see [10], [15]. Thus we define the *future energy function* E_f as

$$E_f(x) := \inf_u \int_0^\infty \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|y(t)\|^2 dt \quad (15)$$

where the infimum is taken over all input functions $u : (0, \infty) \rightarrow \mathbb{R}^m$ taking the system from state x at $t = 0$ to x_0 at time $t = \infty$ (or, more accurately, the controlled system converges for $t \rightarrow \infty$ to x_0).

Because of the second equality in the second line of (10) it follows that

$$\begin{aligned} E_f(x) &= \inf_u [\int_0^\infty (\|v(t)\|^2 - u^T(t)y(t))dt] \\ &= \inf_u [\int_0^\infty \|v(t)\|^2 dt - \int_0^\infty u^T(t)y(t)dt] \\ &= \inf_u [\int_0^\infty \|v(t)\|^2 dt] + H(x) \end{aligned} \quad (16)$$

where the last equality follows from (3) for $t_1 = 0$ and $t_2 = \infty$ together with $x(\infty) = x_0$ and $H(x_0) = 0$.

This last minimization has the obvious solution u being such that $v = 0$, leading to the equality $E_f(x) = H(x)$. (Note that by Assumption 2.1 x_0 is globally asymptotically stable for Σ_s with $v = 0$.)

Secondly we define the *past energy function* E_p as

$$E_p(x) := \inf_u \int_{-\infty}^0 \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|y(t)\|^2 dt \quad (17)$$

where the infimum is taken over all input functions $u : (-\infty, 0) \rightarrow \mathbb{R}^m$ taking the system from state x_0 at $t = -\infty$ to x at time $t = 0$ (or, more accurately, the time-reversed controlled system starting from x at time $t = 0$ converges for $t \rightarrow -\infty$ to x_0).

Because of the first equality in the second line of (10) it follows that

$$\begin{aligned} E_p(x) &= \inf_u \left[\int_{-\infty}^0 (\|z(t)\|^2 + u^T(t)y(t)) dt \right] \\ &= \inf_u \left[\int_{-\infty}^0 \|z(t)\|^2 dt + \int_{-\infty}^0 u^T(t)y(t) dt \right] \\ &= \inf_u \left[\int_{-\infty}^0 \|z(t)\|^2 dt \right] + H(x) \end{aligned} \quad (18)$$

where the last equality follows from (3) for $t_1 = -\infty$ and $t_2 = 0$ together with $x(-\infty) = x_0$ while $H(x_0) = 0$.

This last minimization has the obvious solution u being such that $z = 0$, leading to the equality $E_p(x) = H(x)$. (Note that by Assumption 2.1 x_0 is globally asymptotically stable for the time-reversed Σ_s with $z = 0$.)

In conclusion, both the future and past energies E_f and E_p are equal to H :

$$E_f = H = E_p \quad (19)$$

Nonlinear 'LQG-balancing' is based on comparing the future and past energies [9], [10]. However, because of (19) these two functions are equal to each other. Therefore no information is obtained about the relative importance of the state components.

Remark 3.1: When specialized to a linear lossless system (5) the above result amounts to the fact that the stabilizing solution P to the Control Algebraic Riccati Equation (CARE)

$$A^T P + P A + C^T C - P B B^T P = 0$$

and the inverse of the stabilizing solution S to the Filter Algebraic Riccati Equation (FARE)

$$A S + S A^T + B B^T - S C^T C S = 0$$

are both equal to Q , because Q satisfies (6). In particular the LQG similarity invariants are all to 1, cf. [6], [15], [10].

The same result follows if we apply the recently proposed procedure of *positive real balancing* [2], [14] to Σ . Indeed, when generalized to the nonlinear case, positive real balancing of a passive system is based on comparing the available storage function S_a and the required supply function S_r of Σ . The available storage $S_a(x)$ at x is given as [16], [11]

$$S_a(x) = \sup_{u, T \geq 0} - \int_0^T u^T(t)y(t) dt \quad (20)$$

while the required supply $S_r(x)$ to reach x at $t = 0$ starting from x_0 (assuming reachability from x_0) equals [16], [11]

$$S_r(x) = \inf_{u, T \geq 0, x(-T) = x_0} \int_{-T}^0 u^T(t)y(t) dt \quad (21)$$

It follows from passivity that $S_r(x_0) = 0$, while also $S_a(x_0) = 0$. Furthermore [16], [11], as we will also see in the next section

$$S_a(x) \leq H(x) \leq S_r(x)$$

for all x . In fact, it follows that $S_a \leq S \leq S_r$ for *all* storage functions S , and S_a is the *minimal* and S_r the *maximal* storage function [16], [11].

However, for a lossless system the functions S_a and S_r are immediately seen to be equal [16], and thus

$$S_a = H = S_r \quad (22)$$

B. The scattering representation

Let us now switch attention to the scattering representation Σ_s defined in (9), satisfying the property (11). By our standing assumption the equilibrium x_0 is a globally asymptotically stable equilibrium of Σ_s for $v = 0$. Hence we can apply standard nonlinear open-loop balancing [9]. This involves the computation of the *observability function*

$$O(x) := \int_0^\infty \frac{1}{2} \|z(t)\|^2 dt \quad (23)$$

where $v = 0$ and the integral is taken with initial condition $x(0) = x$. Note that because $\frac{1}{2} \|z(t)\|^2$ is the outgoing power, the observability function $O(x)$ in this case equals the *outgoing (physical) energy*. Since Σ_s is lossless it immediately follows from (11) and $H(x_0) = 0$ that $O(x) = H(x)$.

Secondly, open-loop balancing involves the computation of the *controllability function*

$$C(x) := \inf_v \int_{-\infty}^0 \frac{1}{2} \|v(t)\|^2 dt \quad (24)$$

where the infimum is taken over all input functions $v : (-\infty, 0) \rightarrow \mathbb{R}^m$ taking the state from x_0 at $t = -\infty$ to x at $t = 0$. Thus $C(x)$ in this case is the minimal (physical) energy that is needed to transfer the state from x_0 to x . Applying the first line of (10) and (3) together with $H(x_0) = 0$ we see that

$$\begin{aligned} C(x) &= \inf_v \left[\int_{-\infty}^0 \frac{1}{2} \|z(t)\|^2 dt + \int_{-\infty}^0 u^T(t)y(t) dt \right] \\ &= \inf_v \left[\int_{-\infty}^0 \frac{1}{2} \|z(t)\|^2 dt + H(x) \right], \end{aligned} \quad (25)$$

leading (again using our standing assumption) to the optimal input v being such that $z = 0$, while $C(x) = H(x)$. In fact, we conclude that the minimal energy $\int_{-\infty}^0 \frac{1}{2} \|v(t)\|^2 dt$ to reach x at $t = 0$ is achieved by letting v to be such that the *outgoing* wave vector on $(-\infty, 0)$ is zero. Therefore the minimal input energy is equal to $H(x)$. This is 'dual' to the computation of the observability function for $x(0) = x$, where we already *start* from the assumption that the *incoming* wave v equals zero, resulting in an output energy equal to $H(x)$. (Note that alternatively we could have started from the *minimization* of $\int_0^\infty \frac{1}{2} \|z(t)\|^2 dt$ under the constraint $x(\infty) = x_0$ and *deriving* as the optimal input $v = 0$!)

We conclude that, analogously to (19) and (22)

$$O = H = C \quad (26)$$

Remark 3.2: For a linear lossless system in scattering representation Σ_s given by (14) this amounts to the fact that the observability Gramian M , which is the unique solution to

$$(A - BC)^T M + M(A - BC) = -2C^T C,$$

and the inverse of the controllability Gramian W , which is the unique solution to

$$(A - BC)W + M(A - BC)^T = -2BB^T,$$

are both equal to Q . Hence MW equals the identity matrix, and the Hankel singular values of a linear lossless system in scattering representation are all equal to one.

Hence, like for nonlinear LQG and positive-real balancing for Σ , open-loop balancing for the scattering representation Σ_s does not provide any information about the relative importance of the various state components under our standing assumption.

Finally, we could also apply nonlinear LQG-balancing to the scattering representation Σ_s . However, due to the third line of (10) (the parallelogram identity), the outcome will be the same as nonlinear LQG-balancing for Σ (as analysed before). Indeed, the future and past energy functions E_f and E_p for the scattering representation are *equal* to the future and past energy functions for the power variable representation.

Summarizing we have found the following equalities in the lossless case

$$E_f = O = S_a = H = S_r = C = E_p \quad (27)$$

IV. THE LOSSY CASE

In the previous section we have seen that standard balancing techniques for lossless systems, either in the power variable representation (1) or scattering representation (9), invariably lead to the same result: all state components are equally important. Things do change, however, for passive systems that do have nonzero internal energy-dissipation (sometimes called 'lossy systems').

Recall that the system Σ is called *passive* if the equality in (3) is replaced by an *inequality*, that is

$$H(x(t_2)) - H(x(t_1)) \leq \int_{t_1}^{t_2} u^T(t)y(t)dt \quad (28)$$

for all trajectories $(u(\cdot), x(\cdot), y(\cdot))$ of the system Σ and all time instants $t_1 \leq t_2$. Assuming again differentiability of H this amounts to the Hill-Moylan-Willems conditions [16], [5]

$$\begin{bmatrix} 2 \frac{\partial^T H}{\partial x}(x)a(x) & \frac{\partial^T H}{\partial x}(x) - c^T(x) \\ b^T(x) \frac{\partial H}{\partial x}(x) - c(x) & -d(x) + d^T(x) \end{bmatrix} \leq 0 \quad (29)$$

for all x, u . (These conditions are the generalization of the well-known Kalman-Yakubovich-Popov conditions in the linear case.)

For simplicity of exposition let us assume that $d(x) = 0$ in which case (29) reduces to

$$\begin{aligned} \frac{\partial^T H}{\partial x}(x)a(x) &\leq 0 \\ c(x) &= b^T(x) \frac{\partial H}{\partial x}(x) \end{aligned} \quad (30)$$

Denoting the *dissipated power* $P_d(x) := -\frac{\partial^T H}{\partial x}(x)a(x) \geq 0$ it follows that (28) can be equivalently rewritten as

$$H(x(t_2)) - H(x(t_1)) = \int_{t_1}^{t_2} u^T(t)y(t)dt - \int_{t_1}^{t_2} P_d(x(t))dt$$

for all solution trajectories $(u(\cdot), x(\cdot), y(\cdot))$ of the passive system Σ and all time instants $t_1 \leq t_2$. The last term $\int_{t_1}^{t_2} P_d(x(t))dt$ denotes the internally dissipated energy of the system.

In this case the available storage and the required supply [16] need not be equal to each other, contrary to the lossless case. Indeed, the available storage $S_a(x)$ of Σ is given as [16], [11]

$$\begin{aligned} S_a(x) &= \sup_{u, T \geq 0} - \int_0^T u^T(t)y(t)dt \\ &= H(x) + \sup_{u, T \geq 0} [-H(x(T)) - \int_0^T P_d(x(t))dt] \\ &= H(x) - \inf_{u, T \geq 0} [H(x(T)) + \int_0^T P_d(x(t))dt] \leq H(x) \end{aligned} \quad (31)$$

while the required supply $S_r(x)$ to reach x at $t = 0$ starting from x_0 at $t = -\infty$ equals

$$\begin{aligned} S_r(x) &= \inf_{u, T \geq 0, x(-T)=x_0} \int_{-T}^0 u^T(t)y(t)dt = \\ &= \inf_{u, T \geq 0, x(-T)=x_0} \int_{-T}^0 \frac{d}{dt} H(x(t)) + P_d(x(t))dt = \\ &= H(x) + \inf_{u, T \geq 0, x(-T)=x_0} \int_{-T}^0 P_d(x(t))dt \geq H(x) \end{aligned} \quad (32)$$

Hence, in general there is a *gap* between the available storage S_a and the required supply S_r [16]

$$S_a \leq H \leq S_r \quad (33)$$

and some information about the importance of state components can be gained from balancing these two functions.

On the other hand, the gap in the inequality (33) is critically depending on the amount of internal energy dissipation, resulting e.g. from resistive elements or (in the mechanical domain) damping. In some cases (such as weakly damped mechanical systems), the amount of internal dissipation in the system is difficult to quantify. Hence the outcome of positive real balancing may not be very robust (as opposed to e.g. modal analysis in weakly damped mechanical systems).

Remark 4.1: For a linear passive system Σ given by (5), where for simplicity we assume $D = 0$, the available storage S_a is given as $\frac{1}{2}x^T O_a x$ where Q_a is the *minimal* solution to the Linear Matrix Inequality (LMI) corresponding to (30), namely

$$A^T Q + Q A \leq 0, \quad B^T Q = C \quad (34)$$

while the required supply is $\frac{1}{2}x^T O_r x$ where Q_r is the *maximal* solution to this same LMI.

What happens with the other balancing functions in the lossy case? First, we note that by the first line of (10) with $v = 0$

$$\begin{aligned} O(x) &= \int_0^\infty \frac{1}{2} \|z(t)\|^2 dt = - \int_0^\infty u^T(t)y(t)dt \\ &\leq \sup_{u, T \geq 0} - \int_0^T u^T(t)y(t)dt = S_a(x) \end{aligned} \quad (35)$$

showing that $O \leq S_a$. Furthermore, using the expression for E_f for the scattering representation we trivially obtain

$$\begin{aligned} E_f(x) &= \inf_v [\int_0^\infty \frac{1}{2} \|v(t)\|^2 + \frac{1}{2} \|z(t)\|^2 dt \\ &\leq \int_0^\infty \frac{1}{2} \|z(t)\|^2 dt = O(x) \end{aligned} \quad (36)$$

On the other hand, by making use of the representation of the controllability function C obtained in (25), we have (recall that the infimum is taken over all functions v such that the time-reversed system starting from x at time $t = 0$ converges to x_0 for $t \rightarrow -\infty$)

$$\begin{aligned} C(x) &= \inf_v [\int_{-\infty}^0 \frac{1}{2} \|z(t)\|^2 dt + \int_{-\infty}^0 u^T(t)y(t)dt] \\ &\geq \inf_v \int_{-\infty}^0 u^T(t)y(t)dt \\ &\geq \inf_{u, T \geq 0, x(-T)=x_0} \int_{-T}^0 u^T(t)y(t)dt = S_r(x) \end{aligned} \quad (37)$$

and hence $C \geq S_r$. Furthermore,

$$\begin{aligned} E_p(x) &= \inf_v \int_{-\infty}^0 \frac{1}{2} \|v(t)\|^2 + \frac{1}{2} \|z(t)\|^2 dt \\ &\geq \inf_v \int_{-\infty}^0 \frac{1}{2} \|z(t)\|^2 dt = C(x) \end{aligned} \quad (38)$$

showing that $E_p \geq C$. Collecting all these inequalities we conclude that for any passive system

$$E_f \leq O \leq S_a \leq H \leq S_r \leq C \leq E_p \quad (39)$$

while in the lossless case all inequalities reduce to equalities (27). In general, it would be of interest to investigate when the inequalities in (39) are strict or non-strict.

Remark 4.2: Note that (39) implies that the 'singular values' corresponding to every 'balancing pair' (S_a, S_r) , (O, C) , (E_f, E_p) are all ≤ 1 .

V. PASSIVE SYSTEMS AS PORT-HAMILTONIAN SYSTEMS

In this section we briefly discuss some directions that may be relevant to go beyond the balancing approaches as discussed in the previous sections. For simplicity we restrict ourselves to linear systems.

A first step is the representation of a linear passive system as a linear *port-Hamiltonian system*, given as ([12], [4], [3])

$$\begin{aligned} \dot{x} &= (J - R)Qx + (G - P)u \\ y &= (G + P)^T Qx + (M + S)u, \end{aligned} \quad (40)$$

The Hamiltonian $H(x)$ (the energy of the system) is given by the quadratic function $H(x) = \frac{1}{2}x^T Qx$, where $Q = Q^T$. Furthermore, J is a skew-symmetric $n \times n$ matrix, M is a skew-symmetric $m \times m$ matrix and G is an $n \times m$ matrix, specifying together the *interconnection structure* of the system. The matrices R, S, P , with R a symmetric $n \times n$ matrix, S a symmetric $m \times m$ matrix and P an $n \times m$ matrix, specify the resistive relation, and satisfy the following condition:

$$\begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0 \quad (41)$$

In particular, if $P = 0$, this reduces to the condition that $R \geq 0$ and $S \geq 0$.

The following theorem tells us that any passive linear system can be represented as a port-Hamiltonian system with positive energy function.

Theorem 5.1 (*Passive linear systems are port-Hamiltonian*):

(1). If the linear system (5) is passive with quadratic storage function $\frac{1}{2}x^T Qx$ satisfying $Q \geq 0$, and $Qx = 0$ implies $Ax = 0$ and $Cx = 0$, then (5) can be rewritten into the port-Hamiltonian form (40). (2). If $Q \geq 0$ then the port-Hamiltonian linear system (40) is passive.

Remark 5.1: Note that the condition $(Qx = 0 \Rightarrow Ax = 0, Cx = 0)$ is automatically satisfied if $Q > 0$.

Proof (1) Because of the condition $(Qx = 0 \Rightarrow Ax = 0, Cx = 0)$ it follows from linear algebra that there exists a matrix Σ such that

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} = \Sigma \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \quad (42)$$

Now, passivity of the system (5 with quadratic storage function $\frac{1}{2}x^T Qx$ amounts to the dissipation inequality $x^T Q \dot{x} - u^T y \leq 0$ for all x, u . Substituting $\dot{x} = Ax + Bu$ and $y = Cx + Du$, and making use of (42), this is rewritten as

$$[x^T u^T] \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \Sigma \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0 \quad (43)$$

for all x, u , or equivalently

$$\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} (\Sigma + \Sigma^T) \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \leq 0 \quad (44)$$

It follows from basic linear algebra that we can choose Σ satisfying (42) in such a way that $\Sigma + \Sigma^T \leq 0$. Hence, if we write $\Sigma = \bar{J} - \bar{R}$, $\bar{J} = -\bar{J}^T$, $\bar{R} = \bar{R}^T$ then $\bar{R} \geq 0$. Now, denote

$$\bar{J} = \begin{bmatrix} J & G \\ -G^T & -M \end{bmatrix}, \bar{R} = \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \quad (45)$$

$$J = -J^T, \quad M = -M^T, \quad R = R^T, \quad S = S^T.$$

Then (5) can be written as the port-Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ -y \end{bmatrix} = \left(\begin{bmatrix} J & G \\ -G^T & -M \end{bmatrix} - \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \right) \begin{bmatrix} Qx \\ u \end{bmatrix}, \quad (46)$$

(2) Straightforward computation. \square

The main advantage of the port-Hamiltonian representation of a passive system is that next to the energy-balance as reflected by the passivity property the *interconnection structure* of the system is emphasized. Indeed, while passivity puts only a constraint on the dynamics by relating the increase of internal storage (the increase of energy $\frac{1}{2}x^T Qx$) to the external supply rate (the supplied power $u^T y$), the port-Hamiltonian formulation describes the dynamics by means of the energy storage $\frac{1}{2}x^T Qx$, the resistive relation specified by (41), and the *interconnection structure* of the system. In particular, in the lossless case the dynamics is described by the energy storage $\frac{1}{2}x^T Qx$ and the skew-symmetric matrices J, M , as well as G , while R, P and S are all zero.

VI. STRUCTURE-PRESERVING BALANCING

In previous sections we have seen that balancing of lossless systems basically reduces to transforming the energy $\frac{1}{2}x^T Q x$ into $\frac{1}{2}\tilde{x}^T \tilde{x}$, thus providing no information about the relative importance of the components of the balanced state vector \tilde{x} .

A main idea to go beyond standard balancing is to take into account the interconnection structure of the system, as made explicit in the port-Hamiltonian representation. As a special case let us consider a lossless port-Hamiltonian system *without* inputs and outputs, given by

$$\dot{x} = JQx \quad (47)$$

Furthermore, let us consider the special case (as often encountered in mechanics) that the skew-symmetric matrix J is given as

$$J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} \quad (48)$$

with I_k the $k \times k$ identity matrix (where $2k = n$), and that the energy $\frac{1}{2}x^T Q x$ splits into a *potential energy* and a *kinetic energy*, that is, with $x = \begin{bmatrix} q \\ p \end{bmatrix}$

$$\frac{1}{2}x^T Q x = \frac{1}{2}q^T \bar{Q} q + \frac{1}{2}p^T \bar{P} p \quad (49)$$

with \bar{Q} and \bar{P} both positive definite matrices.

Now let us diagonalize the total energy $\frac{1}{2}x^T Q x$, where we *additionally impose the condition that the diagonalizing transformation leaves the interconnection structure given by the skew-symmetric matrix J invariant*. Since the energy splits into a potential and kinetic energy as in (49) it follows that we may restrict to transformation matrices of the form $W := \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$ for invertible $k \times k$ matrices U, V . Then the invariance condition $WJW^T = J$, with J being given by (48), immediately leads to the condition $V = U^{-T}$. Therefore the transformation of the total energy given by (49) is specified by the following transformation of the matrices \bar{Q} and \bar{P}

$$\bar{Q} \rightarrow U^T \bar{Q} U, \quad \bar{P} \rightarrow U^{-1} \bar{P} U^{-T} \quad (50)$$

It is well-known (see e.g. [1]) that there exists an invertible matrix U such that under the transformation (50) both \bar{Q} and \bar{P} are transformed into diagonal form. In fact, this is precisely the transformation underlying *modal analysis*, see e.g. [7], where simultaneously the kinetic and potential energies are being transformed into diagonal form, with the elements on the diagonal determining the frequencies of the different modes (and where usually model reduction is performed by leaving out the state components corresponding to high frequencies). We conclude that in this case the diagonalization of the energy matrix Q under the additional condition of invariance of the interconnection structure leads to a *non-trivial* diagonalization (that is, Q is *not* transformed into the identity matrix), reflecting the eigenfrequencies and modes of the system without external ports.

The extension of this case to lossless systems *with* inputs and outputs and J again being given by (48) is currently under investigation.

VII. CONCLUSIONS

The general results concerning balancing of lossless or passive systems as derived in Sections III, IV, motivate the ongoing research briefly discussed in Section V. employing the port-Hamiltonian structure, see also [13] for a different approach. In the lossy case the inequalities (39) and their implications for model reduction need more investigation.

A further issue concerns model reduction of passive or port-Hamiltonian systems, which is *structure-preserving*, that is, retains the port-Hamiltonian form. This is especially desirable in network models of complex systems, where one would like to be able to take out high-dimensional components, approximate them by low-dimensional models with the same structure, and then put them back again into the network model.

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